RELATIONS AMONG SMOOTH INTEGRAL MODELS ASSOCIATED TO QUADRATIC, SYMPLECTIC AND HERMITIAN LATTICES

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Abstract. This work is motivated by an investigation into whether, and if so how, certain well known facts about Lie groups manifest in the context of group schemes over rings of integers of local fields. There are the following well-known relations among unitary, orthogonal and symplectic groups:

$$\mathrm{U}(n,\mathbb{R}) = \mathrm{O}(2n,\mathbb{R}) \cap \mathrm{GL}(n,\mathbb{C}) = \mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{GL}(n,\mathbb{C}).$$

Therefore, it is natural to ask whether or not there exist such relations among smooth integral models of unitary, orthogonal and symplectic groups defined over a local field. Moreover, if there do not exist such relations, it would still be worthwhile if one can identify the properties of a hermitian form that lead to failure. We answer all these questions in this paper.

1. Introduction

This paper is motivated by an investigation into whether, and if so how, certain well known facts about Lie groups manifest in the context of group schemes over rings of integers of non-Archimedean local fields.

If h is a nondegenerate hermitian form on a complex vector space, then the real part b_1 of h is a symmetric form on the underlying real space, and the imaginary part b_2 of h is a skew-symmetric real form. In addition, they are related by $b_1(iv, w) = b_2(v, w)$. These facts induce the following well-known relations among unitary, orthogonal and symplectic groups:

$$\mathrm{U}(n,\mathbb{R}) = \mathrm{O}(2n,\mathbb{R}) \cap \mathrm{GL}(n,\mathbb{C}) = \{g \in \mathrm{O}(2n,\mathbb{R}) : J \cdot g = g \cdot J\}$$

and

$$\mathrm{U}(n,\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{GL}(n,\mathbb{C}) = \{g \in \mathrm{Sp}(2n,\mathbb{R}) : J \cdot g = g \cdot J\}.$$

Here, J is a complex structure. Furthermore, if we replace $\mathbb{R}$ and $\mathbb{C}$ by F and K, where F is a non-Archimedean local field and K is a quadratic field extension of F, the above relations remain true. Therefore, it is natural to ask whether or not there exist such relations among smooth integral models of unitary, orthogonal and symplectic groups defined over a local field. Moreover,

if there do not exist such relations, it would still be worthwhile if one can identify the properties of a hermitian form that lead to failure.

In this paper, we answer all these questions. Specifically, Section 5 describes relations between smooth models associated to appropriately related unitary and orthogonal groups, while Section 6 does the same in the context of unitary and symplectic groups.

2. Notations

Notations and definitions in this section are taken from [6] and [3].

- Let F be a non-Archimedean local field with A its ring of integers and κ its residue field.
- Let K be a quadratic field extension of F with B its ring of integers.
- Let p be the residue characteristic of F.
- Let α be an element of B such that $B = A[\alpha]$.
- Let σ be the non-trivial element of the Galois group $\mathrm{Gal}(K/F)$.
- Say $\beta = \alpha \cdot \sigma(\alpha)$.
- We consider a B-lattice L with a hermitian form

$$h : L \times L \to B,$$

  where $h(a \cdot v, b \cdot w) = \sigma(a) b \cdot h(v, w)$ and $h(w, v) = \sigma(h(v, w))$. We denote by a pair (L, h) a hermitian lattice. Assume that $V = L \otimes_A F$ is nondegenerate with respect to h.
- Let G be the reductive algebraic group over F such that

$$G(R) = \mathrm{Aut}_{K \otimes_F R}(V \otimes_F R, h \otimes_F R)$$

  for any commutative F-algebra R.
- Let $\underline{G}'$ be a naive integral model of G such that for any commutative A-algebra R,

$$\underline{G}'(R) = \mathrm{Aut}_{B \otimes_A R}(L \otimes_A R, h \otimes_A R).$$

- Let $\underline{G}$ be the smooth group scheme model of G such that

$$\underline{G}(R) = \underline{G}'(R)$$

  for any étale A-algebra R. Notice that $\underline{G}$ is uniquely determined with these properties by Proposition 3.7 in [6].

From now on, the pair (L, h) is fixed throughout this paper.

3. Description of G and $\underline{G}$

For any $v, w \in L$, we write $h(v,w) = b_1(v,w) + \alpha \cdot b_2(v,w)$ where $b_i(v,w) \in A$ for $i = 1, 2$. In general, b_1 is A-bilinear but not necessarily symmetric. From this section until Section 5, we assume that b_1 is a symmetric bilinear form defined over the F-vector space V. Under this assumption, it is easy to see that $\sigma(\alpha) = -\alpha$. Conversely, in order that b_1 is symmetric, α should satisfy the condition $\sigma(\alpha) = -\alpha$. Let q be the quadratic form on the A-lattice L associated to the symmetric bilinear form b_1, so that

$$q(v) = b_1(v, v).$$

Then $V = L \otimes_A F$ is nondegenerate with respect to q as well as b_1. Notice that $b_2(v,w)$ is a nonsingular alternating bilinear form defined over the F-vector space V and is characterized by

$$b_2(v,w) = \frac{1}{\beta} b_1(\alpha v, w).$$

On the other hand, the element $\alpha \in B$ defines the A-linear homomorphism

$$\widetilde{\alpha} : L \longrightarrow L, v \mapsto \alpha v.$$

Based on this, the Weil restriction of scalars $\mathrm{Res}_{K/F}\mathrm{GL}_K(L \otimes_B K)$ is isomorphic to the closed subgroup scheme of $\mathrm{GL}_F(V)$ satisfying the equation $m \cdot \widetilde{\alpha} = \widetilde{\alpha} \cdot m$, where $m \in \mathrm{GL}_F(V)(R)$ for any commutative F-algebra R. If we consider $\mathrm{O}(V,q)$ and $\mathrm{Res}_{K/F}\mathrm{GL}_K(L \otimes_B K)$ as closed subgroup schemes of $\mathrm{GL}_F(V)$, then the algebraic group G can be identified as follows:

$$G = \mathrm{O}(V,q) \cap \mathrm{Res}_{K/F}\mathrm{GL}_K(L \otimes_B K) = \{g \in \mathrm{O}(V,q) : \widetilde{\alpha} \cdot g = g \cdot \tilde{\alpha}\}.$$

Similarly, by considering $\mathrm{Aut}_A(L,q)$ and $\mathrm{Res}_{B/A}\mathrm{GL}_B(L)$ as closed subgroup schemes of $\mathrm{Aut}_A(L)$, a naive integral model $\underline{G}'$ can be identified as follows:

$$\underline{G}'(R) = \mathrm{Aut}_R(L \otimes_A R, q \otimes_A R) \cap \mathrm{Res}_{B/A}\mathrm{GL}_B(L)(R) = \{g \in \mathrm{Aut}_R(L \otimes_A R, q \otimes_A R) : \widetilde{\alpha} \cdot g = g \cdot \tilde{\alpha}\}$$

for any flat A-algebra R. Therefore, $\underline{G}$ is the smooth group scheme model of $\mathrm{O}(V,q) \cap \mathrm{Res}_{K/F}\mathrm{GL}_K(L \otimes_B K)$ such that

$$\underline{G}(R) = \mathrm{Aut}_R(L \otimes_A R, q \otimes_A R) \cap \mathrm{Res}_{B/A}\mathrm{GL}_B(L)(R) = \{g \in \mathrm{Aut}_R(L \otimes_A R, q \otimes_A R) : \widetilde{\alpha} \cdot g = g \cdot \tilde{\alpha}\}$$

for any étale A-algebra R.

Let $\underline{G_q}'$ be a naive integral model of the orthogonal group $\mathrm{O}(V,q)$, where $V = L\otimes_A F$, such that for any commutative A-algebra R,

$$\underline{G_q}'(R) = \mathrm{Aut}_R(L\otimes_A R, q\otimes_A R).$$

Clearly, there exists a morphism $\underline{G}' \to \underline{G_q}'$. Thus, by functoriality of smooth integral models, this morphism gives the morphism $\underline{G} \to \underline{G_q}$, where $\underline{G_q}$ is the smooth model associated to $\underline{G_q}'$.

4. Review : Construction of Smooth models associated to Quadratic lattice

In this section, we review the construction of a smooth model associated to a quadratic lattice. Our exposition follows that of [6] and [2]. Choose a quadratic A-lattice (L,q) and denote by l the rank of A-lattice L. Assume that $p\neq 2$ ([6]), or that $p=2$ with F an unramified finite extension of $\mathbb{Q}_2$ ([2]).

According to the papers [6] and [2], the first step of the construction of a smooth integral model is to find a suitable functor $\underline{M}$ from the category of commutative A-algebras to the category of monoids. The functor $\underline{M}$ is represented by a unique flat algebra $A(\underline{M})$ which is a polynomial ring over A of l^2 variables (Section 5.2 in [6] and Section 3.2 in [2]). Then the functor $R\mapsto \underline{M}(R)^*$ is represented by the group scheme $\underline{M}^*$ which is an open subscheme of $\underline{M}$. The next step is to construct a functor $\underline{Q}$ such that $\underline{Q}(R)$ is the set of quadratic forms on $L\otimes_A R$ satisfying certain conditions for any flat A-algebra R (Section 5.4 in [6] and Section 3.3 in [2]). The functor $\underline{Q}$ is represented by a polynomial ring over A of $l(l+1)/2$ variables. Note that the fixed quadratic form q is an element of $\underline{Q}(A)$.

Then the group $\underline{M}^*(R)$ acts on the right of $\underline{Q}(R)$ by $f\circ m = {}^t m\cdot f\cdot m$, where $f\in \underline{Q}(R)$ and $m\in \underline{M}^*(R)$, and this action is represented by an action morphism $\underline{Q}\times\underline{M}^* \longrightarrow \underline{Q}$ (Section 5.4 in [6] and Theorem 3.4 in [2]).

Finally, the morphism $\rho : \underline{M}^*\to\underline{Q}$ defined by $\rho(m) = q\circ m$ is smooth and the smooth model $\underline{G_q}$ is the stabilizer of q in $\underline{M}^*$ (Theorem 5.5 in [6] and Theorem 3.5 in [2]).

The proof of smoothness of the morphism ρ is reduced to prove the following lemma (by Theorem 5.5 in [6]):

Lemma 4.1. *(Lemma 5.5.2 in* [6] *and Lemma 3.6 in* [2]*) The morphism* $\rho\otimes\kappa : \underline{M}^*\otimes\kappa \to \underline{Q}\otimes\kappa$ *is smooth.*

We explain an outline of the proof of the lemma. We may assume that $\kappa = \bar{\kappa}$ is algebraically closed. Then, it suffices to show that, for any $m \in (\underline{M}')^*(\bar{\kappa})$, the induced map on the Zariski tangent space $\rho_{*,m} : T_m \to T_{\rho(m)}$ is surjective.

Define two functors from the category of commutative flat A-algebras to the category of abelian groups

$$T_1(R) = \{m - 1 : m \in \underline{M}(R)\},$$

$$T_2(R) = \{f - q : f \in \underline{Q}(R)\}.$$

The functor T_1 (resp. T_2) is representable by a flat A-algebra which is a polynomial ring over A of l^2 (resp. $l(l+1)/2$) variables and they have the structure of a commutative group scheme by showing that they are closed under addition.

In addition, define a suitable functor T_3 represented by a polynomial ring over A of l^2 variables.

Then we identify T_m with $T_1(\bar{\kappa})$ and $T_{\rho(m)}$ with $T_2(\bar{\kappa})$. The map $\rho_{*,m} : T_m \to T_{\rho(m)}$ is then $X \mapsto m^t \cdot q \cdot X + X^t \cdot q \cdot m$, where the sum and the multiplication are to be interpreted as in Section 5.3 of [6].

The proof of this lemma follows by verifying the next three statements:

(1) $X \mapsto q \cdot X$ is a bijection $T_1(\bar{\kappa}) \to T_3(\bar{\kappa})$;
(2) for any $m \in \underline{M}^*(\bar{\kappa})$, $Y \mapsto {}^t m \cdot Y$ is a bijection from $T_3(\bar{\kappa})$ to itself;
(3) $Y \mapsto {}^t Y + Y$ is a surjection $T_3(\bar{\kappa}) \to T_2(\bar{\kappa})$.

5. Main result-hermitian and quadratic lattices

In this section, we state the first main result which describes relations of smooth models associated to a hermitian lattice and a quadratic lattice. Assume that b_1 is symmetric. Recall that the quadratic form q is characterized by $q(v) = b_1(v, v)$. Let n be the rank of B-lattice L.

Assume that the smooth model $\underline{G_q}$ associated to (L, q) is constructed using the method explained in Section 4. That is, we assume that $\underline{G_q}$ is constructed based on $\underline{M}, \underline{M}^*, \underline{Q}, \rho, T_1, T_2, T_3$. For instance, it is constructed explicitly when $p \neq 2$ ([6]) or when $p = 2$ with F an unramified finite extension of $\mathbb{Q}_2$ ([2]).

We define the functor $\underline{M}'$ from the category of commutative flat A-algebras to the category of monoids as follows. For any commutative flat A-algebra R, set

$$\underline{M}'(R) = \{m \in \underline{M}(R) : m \cdot \widetilde{\alpha} = \widetilde{\alpha} \cdot m\}.$$

The functor $\underline{M}'$ is representable by a unique flat A-algebra $A(\underline{M}')$ which is a polynomial ring over A because the functor $\underline{M}$ is representable by a polynomial ring over A. Moreover, the relative dimension of $\underline{M}'$ over Spec A is the same as that of its generic fiber. To compute the dimension of the generic fiber of $\underline{M}'$, let $(e_1, \cdots, e_n)$ be a basis of the given B-lattice L. Then choose a basis $(e_1, \cdots, e_n, \alpha \cdot e_1, \cdots, \alpha \cdot e_n)$ of the A-lattice L. With this choice of basis,

$$\widetilde{\alpha} = \begin{pmatrix} 0 & -\beta \cdot \mathrm{id_n} \\ \mathrm{id_n} & 0 \end{pmatrix},$$

where $\mathrm{id_n}$ is the identity matrix of size $(n \times n)$. Let m be an element of $\underline{M}(R)$ for an F-algebra R. We express the element m by a matrix with respect to the basis chosen above as follows:

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c, d are $(n \times n)$-matrices. Then m satisfies the following easy equations:

$$a = d, b + \beta \cdot c = 0.$$

Therefore, the relative dimension of $\underline{M}'$ over Spec A is $2n^2$.
$\underline{M}'$ has the structure of a scheme of monoids by showing that $\underline{M}'(R)$ is closed under multiplication. Then the functor $R \mapsto \underline{M}'(R)^*$ is represented by a group scheme $(\underline{M}')^*$ and $(\underline{M}')^*$ is an open subscheme of $\underline{M}'$ and is smooth over A.

We define the functor $\underline{Q}'$ from the category of commutative flat A-algebras to the category of monoids as follows. For any commutative flat A-algebra R, set

$$\underline{Q}'(R) = \{f \in \underline{Q}(R) : f \circ \widetilde{\alpha} = \beta \cdot f\}.$$

Here, $f \circ \widetilde{\alpha}(v) = f(\widetilde{\alpha}(v))$. Then the functor $\underline{Q}'$ is represented by a flat A-scheme which is a polynomial ring over A since $\underline{Q}$ is represented by a polynomial ring over A. The dimension of the affine space is the same as that of its generic fiber. To compute its dimension, express f by a matrix with respect to the above basis as follows:

$$f = \begin{pmatrix} x & y \\ {}^t y & z \end{pmatrix},$$

where x, y, z are $(n \times n)$-matrices and x, z are symmetric. Here, ${}^t y$ means the transpose of the matrix y. Then f satisfies ${}^t\widetilde{\alpha} \cdot f \cdot \widetilde{\alpha} = \beta \cdot f$ and this induces the following easy equations:

$$\beta \cdot x = z, y + {}^t y = 0.$$

Therefore, the relative dimension of $\underline{Q'}$ over Spec A is n^2. Note that our fixed quadratic form q is an element of $\underline{Q'}(A)$.

Then for any flat A-algrbra R, the group $(\underline{M'})^*(R)$ acts on the right of $\underline{Q'}(R)$ by $f\circ m = {}^tm\cdot f\cdot m$ and this action is represented by an action morphism

$$\underline{Q'} \times (\underline{M'})^* \longrightarrow \underline{Q'}.$$

We now state the following conjecture.

Conjecture 5.1. *Let ρ' be the morphism $(\underline{M'})^* \to \underline{Q'}$ defined by $\rho'(m) = q \circ m$. Then ρ' is smooth of relative dimension dim G.*

Under the assumption that the conjecture is true, the following corollary is obvious.

Corollary 5.2. *Assume that the above conjecture is true. That is, ρ' is smooth of relative dimension dim G. If we denote by $\underline{G}$ the stabilizer of q in $(\underline{M'})^*$, then the group scheme $\underline{G}$ is smooth, and*

$$\underline{G}(R) = \mathrm{Aut}_R(L \otimes_A R, q \otimes_A R) \cap \mathrm{Res}_{B/A}\mathrm{GL}_B(L)(R)$$

for any étale A-algebra R. Moreover,

$$\underline{G} = \underline{G_q} \cap (\underline{M'})^*$$

and

$$\underline{G}(R) = \{g \in \underline{G_q}(R) : \widetilde{\alpha} \cdot g = g \cdot \widetilde{\alpha}\}$$

for any flat A-algebra R. Here, we are considering $\underline{G_q}$ and $(\underline{M'})^*$ as closed subschemes of $\underline{M}^*$.

The conjecture is true for the cases where the smooth model $\underline{G_q}$ associated to q is constructed in literature ([6] and [2]). We prove this for such cases below.

Theorem 5.3. *Assume that $p = 2$. The conjecture is true if F is an unramified finite extension of $\mathbb{Q}_2$. Notice that b_1 cannot be symmetric if K/F is unramified.*

Proof. Based on Theorem 3.5 in [2], it suffices to show that the morphism $\rho' \otimes \kappa : (\underline{M'})^* \otimes \kappa \to \underline{Q'} \otimes \kappa$ is smooth of relative dimension dim G.

We use the functors T_1, T_2, T_3 defined in Lemma 3.6 of [2]. We define three functors from the category of commutative flat A-algebras to the category of abelian groups as follows:

$$T_1'(R) = \{m - 1 : m \in \underline{M'}(R)\} = \{m - 1 \in T_1(R) : \widetilde{\alpha} \cdot (m - 1) = (m - 1) \cdot \widetilde{\alpha}\},$$

$$T_2'(R)=\{f-q : f\in \underline{Q}'(R)\}=\{f-q\in T_2(R) : (f-q)\circ\widetilde{\alpha}=\beta\cdot(f-q)\}.$$

$$T_3'(R)=\{y\in T_3(R) : {}^t\widetilde{\alpha}\cdot y\cdot\widetilde{\alpha}=\beta\cdot y\}.$$

These three functors are representable by flat A-algebras which are polynomial rings over A of $2n^2, n^2, 2n^2$ variables, respectively, and they have the structure of commutative group schemes by showing that they are closed under addition.

Then, based on the proof of Lemma 4.1, it suffices to show the following three statements:

(1) $X\mapsto q\cdot X$ is a bijection $T_1'(\bar{\kappa})\rightarrow T_3'(\bar{\kappa})$;

(2) for any $m\in(\underline{M}')^*(\bar{\kappa})$, $Y\mapsto {}^tm\cdot Y$ is a bijection from $T_3'(\bar{\kappa})$ to itself;

(3) $Y\mapsto {}^tY+Y$ is a surjection $T_3'(\bar{\kappa})\rightarrow T_2'(\bar{\kappa})$.

For (1), we first observe that two functors T_1' and T_3' are representable by flat affine schemes. Therefore, it suffices to show that the map

$$T_1'(R)\longrightarrow T_3'(R), X\mapsto q\cdot X$$

is bijective for a flat A-algebra R. To prove this, it suffices to show that the map $T_1'(R)\rightarrow T_3'(R), X\mapsto q\cdot X$ and the map $T_3'(R)\rightarrow T_1'(R), Y\mapsto q^{-1}\cdot Y$ are well-defined for all flat A-algebra R.

For the first map, it suffices to show that ${}^t\widetilde{\alpha}\cdot(q\cdot X)\cdot\widetilde{\alpha}=\beta\cdot(q\cdot X)$ because $q\cdot X\in T_3(R)$. This can be proved easily by using definitions of $T_1', \underline{Q}, T_3'$.

Similarly, it can be proved that the second map is well-defined as well.

For (2), it suffices to show that the map

$$T_3'(\bar{\kappa})\longrightarrow T_3'(\bar{\kappa}), Y\mapsto {}^tm\cdot Y, \text{ for any } m\in(\underline{M}')^*(\bar{\kappa}),$$

is well-defined so that its inverse map $Y\mapsto {}^tm^{-1}\cdot Y$ is well-defined as well. Notice that for any flat A-algebra R, the group $\underline{M}^*(R)$ acts on the right of $T_3(R)$ by $Y\circ m={}^tm\cdot Y$ (in the proof of Lemma 3.6 in [2]) and this action is represented by an action morphism

$$T_3\times\underline{M}^*\longrightarrow T_3.$$

Moreover, for elements $m\in(\underline{M}')^*(R)$ and $Y\in T_3'(R)$, where R is a flat A-algebra, we have the following equation:

$${}^t\widetilde{\alpha}\cdot({}^tm\cdot Y)\cdot\widetilde{\alpha}=\beta\cdot({}^tm\cdot Y).$$

Therefore, the above action morphism induces the following action morphism

$$T_3'\times(\underline{M}')^*\longrightarrow T_3'.$$

This proves that the desired map is well-defined.

For (3), we first notice that the dimension of $T_3'(\bar{\kappa})$ is $2n^2$ and that of $T_2'(\bar{\kappa})$ is n^2. In addition, the map $T_3'(\bar{\kappa}) \to T_2'(\bar{\kappa})$ is $\bar{\kappa}$-linear. Therefore, it suffices to show that the dimension of the kernel of the morphism $T_3'(\bar{\kappa}) \to T_2'(\bar{\kappa})$ is n^2.

Define a functor T_4' from the category of commutative A-algebras to the category of abelian groups as follows:

$$T_4'(R) = \textit{Kernel of the map} : T_3'(R) \longrightarrow T_2'(R), Y \mapsto {}^tY + Y.$$

Let $n_i = \mathrm{rank}_B L_i$ and so $n = \mathrm{rank}_B L = \sum n_i$. We choose a Jordan splitting $L = \bigoplus_i L_i$ and a basis of (L, h) as explained in Theorem 2.8 of [3]. Let $(e_1, \cdots, e_{n_i})$ be a basis of the B-lattice L_i. Then choose a basis $(e_1, \cdots, e_{n_i}, \alpha e_1, \cdots, \alpha e_{n_i})$ of the A-lattice L_i. With respect to this choice of basis,

$$\widetilde{\alpha}_i := \widetilde{\alpha}|_{L_i} = \begin{pmatrix} 0 & -\beta \cdot \mathrm{id_{n_i}} \\ \mathrm{id_{n_i}} & 0 \end{pmatrix},$$

where $\mathrm{id_{n_i}}$ is the identity matrix of size $(n_i \times n_i)$. Therefore,

$$\widetilde{\alpha} = \left(\widetilde{\alpha}_i\right)$$

with $\widetilde{\alpha}_i$ for the (i,i)-block and 0 for remaining blocks.

Then it is obvious that T_4' is represented by a flat algebra which is a polynomial ring over A as well. Thus the dimension of $T_4'(\bar{\kappa})$ is the same as that of the generic fiber of T_4'. Now it is easy to compute that the latter is n^2. □

Theorem 5.4. *The conjecture is true if $p \neq 2$. Notice that if $p \neq 2$, there exists a suitable choice of α which induces b_1 to be symmetric.*

The proof of this theorem is similar to that of the above theorem and is left to the reader.

Remark 5.5. To summarize,

(1) Assume that $p = 2$. The conjecture is true if F is an unramified finite extension of $\mathbb{Q}_2$. Notice that b_1 is symmetric for an appropriate α if K/F is ramified. However, b_1 cannot be symmetric if K/F is unramified.
(2) The conjecture is true if $p \neq 2$.

6. Main result-hermitian and symplectic lattices

In this section, we state the second main result which describes relations of smooth models associated to a hermitian lattice and a symplectic lattice. We keep the notation from the previous section. Notice that the bilinear form b_2 is always alternating and $V = L \otimes_A F$ is nondegenerate with respect to b_2. We emphasize that we do not require that b_1 is symmetric. The bilinear form b_1 is characterized as follows:

$$b_1(v, w) = -b_2(\alpha v, w).$$

Therefore, the algebraic group G can be identified as follows:

$$G = \mathrm{Sp}(V, b_2) \cap \mathrm{Res}_{K/F}\mathrm{GL}_K(L \otimes_B K).$$

Similarly, we have the following bijection:

$$\underline{G}'(R) = \mathrm{Aut}_R(L \otimes_A R, b_2 \otimes_A R) \cap \mathrm{Res}_{B/A}\mathrm{GL}_B(L)(R) = \{g \in \mathrm{Aut}_R(L \otimes_A R, b_2 \otimes_A R) : \widetilde{\alpha} \cdot g = g \cdot \tilde{\alpha}\}$$

for any flat A-algebra R. Therefore, $\underline{G}$ is the smooth group scheme model of $\mathrm{Sp}(V, b_2) \cap \mathrm{Res}_{K/F}\mathrm{GL}_K(L \otimes_B K)$ such that

$$\underline{G}(R) = \mathrm{Aut}_R(L \otimes_A R, b_2 \otimes_A R) \cap \mathrm{Res}_{B/A}\mathrm{GL}_B(L)(R) = \{g \in \mathrm{Aut}_R(L \otimes_A R, b_2 \otimes_A R) : \widetilde{\alpha} \cdot g = g \cdot \tilde{\alpha}\}$$

for any étale A-algebra R.

Let $\underline{G_{b_2}}'$ be a naive integral model of the symplectic group $\mathrm{Sp}(V, b_2)$, where $V = L \otimes_A F$, such that for any commutative A-algebra R,

$$\underline{G_{b_2}}'(R) = \mathrm{Aut}_R(L \otimes_A R, b_2 \otimes_A R).$$

Clearly, there exists a morphism $\underline{G}' \rightarrow \underline{G_{b_2}}'$. Then, by functoriality of smooth integral models, this morphism gives the morphism $\underline{G} \rightarrow \underline{G_{b_2}}$, where $\underline{G_{b_2}}$ is the smooth model associated to $\underline{G_{b_2}}'$.

The construction of $\underline{G_{b_2}}$ is similar to that of $\underline{G_q}$. For an explicit construction, we refer to Section 5 of [6].

Assume that $\underline{G_{b_2}}$ is constructed based on $\underline{M}, \underline{M}^*, \underline{A}, \rho, T_1, T_2, T_3$ as explained in Section 4. Caution that we take a functor $\underline{A}$, instead of $\underline{Q}$, such that $\underline{A}(R)$ is the set of alternating forms on $L \otimes_A R$ satisfying certain conditions for any flat A-algebra R, as explained in Section 5.2 of [6]. Note that we have treated quadratic lattices in Section 4 and the same method can be applied to construct $\underline{G_{b_2}}$ (Section 5 of [6]).

We define the functors $\underline{M}'$ and $\underline{A}'$ on the category of commutative flat A-algebras as before (Section 5). Namely, for any commutative flat A-algebra R,

$$\underline{M}'(R) = \{m \in \underline{M}(R) : m \cdot \widetilde{\alpha} = \widetilde{\alpha} \cdot m\},$$

$$\underline{A}'(R) = \{f \in \underline{A}(R) : f \circ \widetilde{\alpha} = \beta \cdot f\}.$$

Here, $f \circ \widetilde{\alpha}(v) = f(\widetilde{\alpha}(v))$. Then both of $\underline{M}'$ and $\underline{A}'$ are represented by unique flat A-algebras which are polynomial rings over A. Moreover, the relative dimensions of $\underline{M}'$ and $\underline{A}'$ over Spec A are the same as those of their generic fibers and they are $2n^2$ and n^2, respectively. These dimensions can be computed similarly as in Section 5.

$\underline{M}'$ has the structure of a scheme of monoids by showing that $\underline{M}'(R)$ is closed under multiplication. Then the functor $R \mapsto \underline{M}'(R)^*$ is represented by a group scheme $(\underline{M}')^*$ and $(\underline{M}')^*$ is an open subscheme of $\underline{M}'$ and is smooth over A. Note that our fixed alternating form b_2 is an element of $\underline{A}'(A)$.

Then for any flat A-algrbra R, the group $(\underline{M}')^*(R)$ acts on the right of $\underline{A}'(R)$ by $f \circ m = {}^t m \cdot f \cdot m$ and this action is represented by an action morphism

$$\underline{A}' \times (\underline{M}')^* \longrightarrow \underline{A}'.$$

Our setting is similar to that of Section 5. Thus the following question naturally arises:

Question 6.1. *Let ρ' be the morphism $(\underline{M}')^* \to \underline{A}'$ defined by $\rho'(m) = b_2 \circ m$. Is ρ' smooth of relative dimension dim G?*

In general, the answer to Question 6.1 is no. If K/F is ramified with $p = 2$, this is no longer true. As a counter-example, choose a hermitian lattice L having a Gram matrix $\begin{pmatrix} 1 \end{pmatrix}$. Then it is easily seen that the fiber of b_2 in $(\underline{M}')^*$ under the morphism ρ' is not smooth. This induces that ρ' is not smooth. However, this question is true in other cases.

Theorem 6.2. *The morphism ρ' is smooth of relative dimension dim G if K/F is unramified with $p = 2$.*

Proof. We define three functors from the category of commutative flat A-algebras to the category of abelian groups as follows:

$$T_1'(R) = \{m - 1 : m \in \underline{M}'(R)\} = \{m - 1 \in T_1(R) : \widetilde{\alpha} \cdot (m - 1) = (m - 1) \cdot \widetilde{\alpha}\},$$

$$T_2'(R) = \{f - b_2 : f \in \underline{A}'(R)\} = \{f - b_2 \in T_2(R) : (f - b_2) \circ \widetilde{\alpha} = \beta \cdot (f - b_2)\}.$$

$$T_3'(R) = \{y \in T_3(R) : {}^t\widetilde{\alpha} \cdot y \cdot \widetilde{\alpha} = \beta \cdot y\}.$$

Then it suffices to show the following three statements:

(1) $X \mapsto q \cdot X$ is a bijection $T_1'(\bar{\kappa}) \to T_3'(\bar{\kappa})$;

(2) for any $m \in (\underline{M}')^*(\bar{\kappa})$, $Y \mapsto {}^t m \cdot Y$ is a bijection from $T_3'(\bar{\kappa})$ to itself;

(3) $Y \mapsto {}^tY + Y$ is a surjection $T_3'(\bar{\kappa}) \to T_2'(\bar{\kappa})$.

The rest of the proof is similar to that of Theorem 5.2 and so we will omit it. $\square$

If $p \neq 2$, the answer to Question 6.1 is yes and its proof is similar to the above. We state this as the following theorem:

Theorem 6.3. *The morphism ρ' is smooth of relative dimension dim G if $p \neq 2$.*

To summarize, we have the following theorem.

Theorem 6.4. *(1) Assume that K/F is unramified or that $p \neq 2$. Then, the morphism ρ' is smooth of relative dimension dim G. Let $\underline{G}$ be the stabilizer of b_2 in $(\underline{M}')^*$. The group scheme $\underline{G}$ is then smooth, and*

$$\underline{G}(R) = \mathrm{Aut}_R(L \otimes_A R, b_2 \otimes_A R) \cap \mathrm{Res}_{B/A}\mathrm{GL}_B(L)(R)$$

for any étale A-algebra R. Moreover,

$$\underline{G} = \underline{G_{b_2}} \cap (\underline{M}')^*$$

and

$$\underline{G}(R) = \{g \in \underline{G_{b_2}}(R) : \widetilde{\alpha} \cdot g = g \cdot \widetilde{\alpha}\}$$

for any flat A-algebra R. Here, we consider $\underline{G_{b_2}}$ and $(\underline{M}')^*$ as closed subschemes of $\underline{M}^*$.

(2) If K/F is ramified with $p = 2$, then the morphism ρ' is not necessarily smooth.

Remark 6.5. (1) As we have seen in this section, if K/F is ramified with $p = 2$, then $\underline{G_{b_2}} \cap (\underline{M}')^*$ is no longer smooth. The reason explaining this is the following:

We define the dual lattice of L with respect to b_2, denoted by $L_A^{\perp}$, as follows:

$$L_A^{\perp} = \{x \in L \otimes_A F : b_2(x, L) \subset A\}.$$

Similarly, we define the dual lattice of L with respect to h, denoted by $L_B^{\perp}$, as follows:

$$L_B^{\perp} = \{x \in L \otimes_A F : h(x, L) \subset B\}.$$

Recall that we have assumed that $\underline{G_{b_2}}$ is constructed based on $\underline{M}$. Assume that $\underline{G}$ is constructed based on $\widetilde{\underline{M}}$ ([6] and [3]). Based on Section 5 of [6], $\underline{M}$ is characterized as follows:

$$\underline{M}(R) = \{m \in \mathrm{End}_R(L \otimes_A R) : m(L_A^{\perp} \otimes_A R) \subset L_A^{\perp} \otimes_A R\},$$

for any commutative flat A-algebra R. Then, $\underline{M}'$ is characterized as follows:

$$\underline{M}'(R)\ (= \{m \in \underline{M}(R) : m \cdot \widetilde{\alpha} = \widetilde{\alpha} \cdot m\}) = \{m \in \mathrm{End}_{B\otimes_A R}(L \otimes_A R) : m(L_B^{\perp} \otimes_A R) \subset L_B^{\perp} \otimes_A R\}.$$

Based on [3] and Section 5 of [6], $\widetilde{\underline{M}} = \underline{M}'$ if K/F is unramified or $p \neq 2$. Otherwise, that is, if K/F is ramified with $p = 2$, it is not necessary that $\widetilde{\underline{M}} = \underline{M}'$.

(2) We briefly explain which properties of (L, h) should be reflected in the construction of $\widetilde{\underline{M}}$ when (L, h) is a ramified hermitian lattice with $p = 2$. From now on, we assume that $p = 2$ and that K/F is a ramified quadratic field extension with F an unramified finite extension of $\mathbb{Q}_2$. Assume that a Gram matrix of (L, h) has a unit determinant. (Thus, L is π_B^0-modular. For the definition of a π_B^i-modular lattice, see Definition 2.1 of [3]). Then there exist the following properties of (L, h):

(a) The function

$$q\ mod\ 2 : L/\pi_B L \longrightarrow \kappa, x \mapsto h(x, x)\ mod\ 2$$

is an additive polynomial. This fails when K/F is unramified or $p \neq 2$. Let B_0 be the sublattice of L such that $B_0/\pi_B L$ is the kernel of this function. Then for all element $g \in \mathrm{Aut}_B(L, h)$, g induces the identity on L/B_0.

(b) Assume that the above additive polynomial defined over $L/\pi_B L$ is not trivial. That is, $B_0 \subsetneq L$. Then the bilinear form h mod π_B on the κ-vector space $L/\pi_B L$ is nonsingular symmetric and non-alternating. It is well known that there is a unique vector $e \in L/\pi_B L$ such that $h(v, e)^2 = h(v, v)$ mod π_B for every vector $v \in L/\pi_B L$. Then for all element $g \in \mathrm{Aut}_B(L, h)$, g fixes the vector e.

For an explicit construction of $\widetilde{\underline{M}}$, we refer to [3].

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